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INVARIANT ANISOTROPIC LARGE DEFORMATION DETERMINISTIC AND STOCHASTIC COMBINED LOAD FAILURE CRITERIA

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Abstract—The deterministic Shanley–Ryder interaction failure criterion is reformulated in terms of generalized stress tensors and cast into two distinct invariant forms, one for applied stresses and the other for failure stresses. This generalization extends its use to deterministic and stochastic problems with large deformations as well as to any arbitrary anisotropic material. Additionally, the original N-dimensional failure surfaces corresponding to N combined loads are reduced to a universal three-dimensional failure surface in terms of three loading, geometric and material parameters. Although a greater number of such parameters could be used, the accuracy of fit to data based on only three parameters negates such a need. Illustrative examples are presented and the influence of parametric variations is discussed.

INTRODUCTION

In 1937, Shanley and Ryder proposed the so-called interaction curve stress ratio failure criterion for combined deterministic loads. It essentially relates empirically each specific load to its corresponding uniaxial failure stress and contains sets of parameters which are functions of the types and numbers of combined loads and of material properties as well as geometries. This criterion has withstood the test of time and is the standard failure criterion in the aerospace industry. A wealth of information on parameters in relation to materials, loadings and geometries can be found in the MIL Handbook (1991).

Since its original introduction, this criterion has undergone extensions and modifications to account for time-dependent and/or stochastic failures. Hilton (1952) used this principle to determine critical times for the creep buckling of viscoelastic columns. Yi (1991) extended the deterministic quadratic delamination criterion of Brewer and Lagace (1988) (a form of interaction curves) to cover time-dependent viscoelastic delamination onsets. Hilton (1992) reformulated the deterministic interaction curves to take into account stochastic loads and material properties such as moduli and failure stresses. Finally, Hilton and Yi (1993) generalized the Yi (1991) delamination criterion to cover three-dimensional random loads, geometries and material properties and thus permit probabilistic delamination onset determinations as functions of time to failure.

Cederbaum *et al.* (1989) examined the reliability of laminated plates subjected to random static loads using the Shanley–Ryder interaction curves for two-dimensional tension and compression and unidirectional shear. The inherent difficulties associated with anisotropic behavior are quite apparent as the authors are forced to use four distinct interaction curves for tension and compression of the fibers and of the matrices. Hasofer and Lind (1974) formulated a different invariant second moment failure criterion based on only three separate stress components. Tsai and Wu (1971) devised a failure criterion in terms of multiple specialized stress invariants which do not lend themselves to a three-dimensional visualization and which need to be reformulated for each combined load set. Previously, Gol'denblat and Kopnov (1966) proposed a failure relationship based on invariant combinations of applied and failure stresses, rather than on separate invariants based on each of the two types of stresses as is done in the present study.

However, while only deterministic, none of the above failure analyses consider large deformations nor general material anisotropy, since all are formulated in terms of stresses along undeformed Cartesian axes. In order to remove the small deformation restrictions, one needs to introduce generalized coordinates moving with the deforming body and/or structural components. Anisotropy can best be supported by expressions in terms of stress invariants which are insensitive, except for magnitudes, to the distortions of the body and to its intrinsic (embedded) coordinate shape changes. This invariant large deformation formulation results in a universal three-dimensional deterministic or stochastic failure surface criterion with three or more parameters which are functions of (1) the type and number of loadings, (2) material properties and (3) structural geometry. In this form it can readily be applied in a general fashion to large flexible space structures and to composites where fiber orientations directly influence material anisotropy and where deflections may also be sizeable. On the other hand, the Shanley–Ryder criterion must be re-evaluated for each set of material coordinate directions used in specific anisotropic analyses of each structural member.

ANALYSIS

Consider a body or a part of a structure in a generalized nonorthogonal curvilinear coordinate system θ^i ($i = 1, 2, 3$) subjected to N number of external loads, such as bending, tension, compression, torsion, shear, etc., based on applied stresses σ_n corresponding to each loading and on their associated uniaxial failure stresses F_n in the form of (Shanley and Ryder, 1937)

$$\sum_{n=1}^{N} \left(\frac{\sigma_n}{F_n}\right)^{a_n} = 1 \tag{1}$$

and where the experimentally determined exponents a_n depend on each loading as well as on their combinations, on material properties and on geometry of each structural component. The failure condition of eqn (1) has since been extended to combined stochastic loads and failure stresses by Hilton (1992), and to time-dependent deterministic and stochastic viscoelastic composite delaminations by Yi (1991) and by Hilton and Yi (1993), respectively. Condition (1) as well as the three cited publications all use essentially Cartesian stress components referred to each particular type of loading, thus making them unsuitable for large deformations and awkward, to say the least, for more complicated anisotropic states.

A generalization of these failure criteria is, therefore, proposed in terms of curvilinear Cauchy stress tensors τ^i_j and F^i_j for each type of loading [see Green and Zerna (1954) for the generalized tensor notation]. The three fundamental invariants of the second order tensors τ^i_j and F^i_j can be written as

$$\begin{aligned} J_1 &= \tau^i_i \\ J_2 &= \tau^i_j \tau^j_i \\ J_3 &= \tau^i_j \tau^j_k \tau^k_i \end{aligned} \tag{2}$$

and similarly

$$\mathscr{J}_1 = F^i_i$$

$$\mathscr{J}_2 = F^i_j F^j_i$$

$$\mathscr{J}_3 = F^i_j F^j_k F^k_i, \tag{3}$$

where the usual one third has been omitted since only ratios $J_k/\mathscr{J}_k$ are being used and each τ^i_j and F^i_j is replaced by its absolute value [see comments after eqn (5)]. The generalized tensor summation convention applies in all relations. The Cayley–Hamilton theorem states that there are only three fundamental independent invariants associated with any second order tensor (Brand, 1947). Consequently, any three invariants, such as those of eqns (2) and (3), are sufficient to formulate an invariant failure condition. However, it must be noted that the odd numbered invariants vanish whenever only shear stresses are present in the system, i.e. $\tau^i_j = 0$ for $i = j$. Since a multiplicative invariant failure law is proposed in this paper, should this become a drawback then only even numbered invariants J_k and $\mathscr{J}_k$ should be used. On the other hand, in the illustrative example displayed in the discussion section, only the first three invariants are used without any impingement on the accuracy of the solution.

An invariant failure criterion can then be formed in terms of these invariants paralleling the Shanley–Ryder construction such that for deterministic conditions (stochastic ones will be considered subsequently)

$$\frac{1}{3}\sum_{k=1}^{3}\left(\frac{J_k}{\mathscr{J}_k}\right)^{b_k} = 1. \tag{4}$$

The one third in eqn (4) is introduced since, unlike the criterion (1), the invariant condition (4) always has three terms even for uniaxial loadings when, say, τ^1_1 is the only nonzero stress [see eqns (2) and (3) and subsequent analysis and discussion when $\tau^1_1 = 0$].

Since the failure condition (4) must always be reducible to one-dimensional uniaxial failure cases where $\tau^1_1 = F^1_1$, care must be exercised to properly form the failure stress invariants $\mathscr{J}_k$, which can only contain the F^i_j corresponding to the existing stress states τ^i_j. For instance, a combination of normal and shear stresses τ^1_1 and τ^1_2 has corresponding $\mathscr{J}_k$s for symmetric $F^i_j = F^j_i$:

$$\mathscr{J}_1 = F^1_1$$

$$\mathscr{J}_2 = (F^1_1)^2 + 2(F^1_2)^2$$

$$\mathscr{J}_3 = (F^1_1)^3 + 2(F^1_2)^2 F^1_1. \tag{5}$$

An additional important comment relating to the signs of the invariants J_k and $\mathscr{J}_k$ needs to be entered. In eqn (1), each σ_n has the same algebraic sign as its corresponding failure stress F_n since each pair refers to the same uniaxial condition. However, such is not the case with the three invariants J_k and $\mathscr{J}_k$ since any of them can be either positive, negative or zero without regard to its corresponding pair. Consequently, in order to avoid the pitfalls of J_k or $\mathscr{J}_k \leqslant 0$ resulting in possible zero denominators or negative $J_k/\mathscr{J}_k$ ratios raised to non-integer powers or $J_k/\mathscr{J}_k$ exceeding unity, all τ^i_j and F^i_j components in eqns (2) and (3) are replaced by their absolute values. From a physical point of view, this makes sense since in the above example, for instance, the value of $\mathscr{J}_1$ or $\mathscr{J}_3$ should have the identical influence on failure without regard to the signs of F^1_1 and/or F^1_2. That is to say, $\mathscr{J}_1$ or $\mathscr{J}_3$ should not be diminished when F^1_1 is compressive as compared to a tensile failure stress F^1_1 (except for possible differences in numerical values for these two distinct normal failure stresses). Similarly, a negative F^1_2 should not decrease $\mathscr{J}_3$.

In other loading conditions where unequal tensile and compressive stresses are present on different parts of the same surface in the same direction, as would be due to unsymmetrical bending, or where ultimate stresses in compression and tension differ, invariants for both conditions must be evaluated separately to see which condition is the predominant one and more likely to first result in failure.

It should be noted that the Shanley–Ryder relationship (1) is representable in a N-dimensional space depending on the number N of distinct combined loads, thus making it non-unique and creating pictorial representations which are most difficult, if not impossible. On the other hand, the present failure condition (4), in addition to being invariant, is always representable in a three-dimensional space $J_k/\mathscr{J}_k$ regardless of how many combined loads N are present in the system. The three-dimensional failure surface given by eqn (4) is universal and its specific shape is only governed by the exponents b_k, which depend on types and number of loadings, material properties and on structural geometry.

The invariant failure condition (4) is written as a matter of convenience in a form similar to the original Shanley–Ryder condition (1). Since both equations represent analytical failure surfaces of variables σ_n/F_n and $J_k/\mathscr{J}_k$ each in some finite domain, they can be expressed functionally in power series. In particular, the general, but also more cumbersome, expression for condition (4) is

$$F_{\text{INV}}\left(\frac{J}{\mathscr{J}}\right) = \sum_{k=0}^{K}\sum_{l=0}^{L}\sum_{m=0}^{M} C_{klm}\left(\frac{J_1}{\mathscr{J}_1}\right)^k\left(\frac{J_2}{\mathscr{J}_2}\right)^l\left(\frac{J_3}{\mathscr{J}_3}\right)^m = 1, \tag{6}$$

with $K = L = M = \infty$. From a practical standpoint, one would truncate the three series with some finite, but not necessarily equal, values for K, L and M. Equation (6), no matter what the values of K, L or M, is still representable by a three-dimensional surface.

Similarly, the Shanley–Ryder condition can be generalized to

$$F_{\text{S-R}}\left(\frac{\sigma}{F}\right) = \sum_{k=0}^{K}\sum_{l=0}^{L}\sum_{m=0}^{M}\cdots\sum_{r=0}^{R} D_{klm\ldots r}\left(\frac{\sigma_1}{F_1}\right)^k\left(\frac{\sigma_1}{F_1}\right)^k\left(\frac{\sigma_2}{F_2}\right)^l\left(\frac{\sigma_3}{F_3}\right)^m\cdots\left(\frac{\sigma_N}{F_N}\right)^r = 1, \tag{7}$$

with N representing the number of loading conditions as before in eqn (1) and yielding an N-dimensional failure surface.

In essence then, eqns (1), (4) and (7) represent approximations to the corresponding infinite or truncated series in terms of a limited set of generally non-integer exponents a_n and b_k. Parenthetically, it is worth noting that such expressions as the octahedral shear stress law and the Huber–von Mises–Hencky plasticity condition (Freudenthal, 1950) are among some of the degenerate examples of the general invariant form (6).

From a computational point of view, eqns (1) and (4) are of most awkward forms when determining the exponents a_n or b_k by the least squares method. A much more computationally attractive expression is

$$F_{\text{INV}}\left(\frac{J}{\mathscr{J}}\right) = C\left(\frac{J_1}{\mathscr{J}_1}\right)^{b_1}\left(\frac{J_2}{\mathscr{J}_2}\right)^{b_2}\left(\frac{J_3}{\mathscr{J}_3}\right)^{b_3} = 1 \tag{8}$$

which, when taking logarithms, results in linear relations for the three unknown b_k exponents, thus making their determination from experimental data very simple. These procedures and results will be covered in the next section.

This invariant deterministic failure criterion in the form (4), (6) or (8) can be readily extended to stochastic loads, material properties and failure stresses following the non-invariant random formulation of Hilton (1992). Consider random stresses $\tilde{\tau}_j^i$ with mean values τ_j^i resulting from stochastic combined applied loads, material properties (moduli), temperatures, moisture contents, geometries, etc. Invariants $\tilde{J}_k(\tilde{\tau}_j^i)$ and $J_k(\tau_j^i)$ correspond to these stresses. Similarly, consider random failure stresses $\tilde{F}_j^i$ with mean values F_j^i resulting in invariants $\tilde{\mathscr{J}}_k(\tilde{F}_j^i)$ and $\mathscr{J}_k(F_j^i)$. Two separate stochastic combined load failure criteria can now be constructed, such that

$$F_{\text{INV}}\left(\frac{\tilde{J}}{\mathscr{J}}\right) = \tilde{v} \tag{9}$$

and

$$F_{\text{INV}}\left(\frac{\tilde{\mathscr{J}}}{\mathscr{J}}\right) = \tilde{V}, \tag{10}$$

where the function F_{INV} $(\tilde{\mathscr{J}}/\mathscr{J})$ has an analogous construction to that of F_{INV} $(\tilde{J}/\mathscr{J})$ given by eqns (4), (6) and/or (8). Failure will occur whenever

$$\tilde{u} = \tilde{V} - \tilde{v} \leqslant 0. \tag{11}$$

The random variables $\tilde{v}$ and $\tilde{V}$ each may have distinct probability density distributions $\phi(\tilde{v})$ and $\Phi(\tilde{V})$ with appropriate moments depending on the distributions of $\tilde{J}_k$ and $\tilde{\mathscr{J}}_k$. In fact, as formulated previously by Hilton (1992), the exponents b_k in eqns (9) and (10) are taken as deterministic and all the stochastic properties of the $\tilde{\tau}^i_j$ and $\tilde{F}^i_j$ stresses are totally absorbed in, and prescribed by, the $\phi(\tilde{v})$ and $\Phi(\tilde{V})$ distributions. Similarly, from eqn (11), the variable $\tilde{u}$ has a probability density distribution $\psi(\tilde{u})$ derived from a convolution integral of the two distributions ϕ and Φ (Lin, 1967). The probability of failure $P(u)$ of a structural component under N combined load is

$$P(u) = \int_0^U \psi(z)\mathrm{d}z, \tag{12}$$

with

$$U = u/\sigma \quad \text{and} \quad z = (\tilde{u} - u)/\sigma \tag{13}$$

and where σ^2 is the standard deviation of z and u is the mean of $\tilde{u}$ (Hilton, 1992).

DISCUSSION OF RESULTS

The three exponents b_k in eqns (4), (6) and (8) or the many coefficients C_{klm} in eqn (6) can be generated from either (i) experimental data or (ii) a least square fit of eqn (4) or (8) to the Shanley–Ryder curves of eqn (1).

As a means of illustrating procedure (ii), consider the following deterministic example with normalized stress values for $N = 4$:

$F_1 = 1.0$ tension in x_1 direction

$F_2 = 1.1$ bending tension or compression in x_1 direction

$F_3 = 0.9$ tension in x_2 direction

$F_4 = 0.8$ shear on x_1 plane in x_2 direction

$a_1 = 1.2$ $a_2 = 1.5$ $a_3 = 2.0$ $a_4 = 2.5$

which specifies eqn (1) to be

$$\left(\frac{\sigma_1}{1.0}\right)^{1.2} + \left(\frac{\sigma_2}{1.1}\right)^{1.5} + \left(\frac{\sigma_3}{0.9}\right)^{2.0} + \left(\frac{\sigma_4}{0.8}\right)^{2.5} = 1. \tag{14}$$

The parametric values used here are selected to represent a typical illustrative example and do not necessarily correspond to any specific material. By using the least squares method after first taking the logarithms of eqn (8), the b_k exponents of eqns (8), (9) and (10) can be determined in the following manner. Form a least squares relation between eqn (1), i.e. (14), and eqn (8) such that the sum of errors for a set of P points is

$$E_s = \sum_{p=1}^{P} e_p = \sum_{p=1}^{P} \left[\sum_{k=1}^{3} b_k \log\left(\frac{J_{pk}}{\mathscr{J}_{pk}}\right) + C^* \right]^2 \tag{15}$$

and which will be minimum when each

$$\frac{\partial E_s}{\partial b_r} = 0 \qquad (r = 1, 2, 3) \tag{16}$$

is satisfied or

$$\sum_{p=1}^{P} \left\{ \sum_{k=1}^{3} \left[b_k \log\left(\frac{J_{pk}}{\mathscr{J}_{pk}}\right) + C^* \right] \log\left(\frac{J_{pk}}{\mathscr{J}_{pk}}\right) \right\} = 0 \quad (r = 1, 2, 3) \tag{17}$$

and where $C^* = \log C$.

The set of P points satisfying eqn (1), or specifically in this example of eqn (14), is generated and then used in eqn (17) to solve for the exponents b_k. The invariants are determined from eqns (2) and (3) with τ_j^i and F_j^i corresponding to the absolute values of the set of P points used in eqn (1).

Ideally, the coefficient C in eqn (8) should be unity, however this would then make $C^* = 0$. Under the latter condition eqn (17) then becomes three homogeneous linear algebraic relations for the b_k exponents leading to eigenvalues where physically and mathematically none exist and thereby casting doubt on the validity of eqn (8) for $C = 1$ as an appropriate approximation for the most general expression (8). However, in the approximation (8) a value of C other than unity, i.e. $1 \pm \varepsilon$, may be selected such that the corresponding b_ks give the "best fit" to the data, thus removing any conditions leading to spurious eigenvalues.

Table 1 displays the results of computations to determine a reasonable value of C in eqn (8) to fit the example of eqn (14). It can be readily seen that, in the C range of $1.0 \pm 1.E{-}9$ to $1.0 \pm 1.E{-}11$, extremely good agreement results between the Shanley–Ryder form (14) and the invariant condition (8).

The accuracy of the fit, as indicated by Table 1, for an arbitrary but realistic Shanley–Ryder interaction curve example of eqn (14), shows that the three parameters of the invariant failure surface (8) are sufficient to adequately characterize failure conditions. However, if desired or necessary, the fuller complement of $K+L+M$ parameters C_{klm} of eqn (6) is always available for a least square fit of any data set.

Table 1. Relation of C in eqn (8) to accuracy of eqn (15) fit for first three invariants

C	Maximum fit error	σ^2
1.0±1.E−1	0.3722E−1	0.2025E−3
1.0±1.E−3	0.3471E−3	0.1821E−7
1.0±1.E−5	0.3469E−5	0.1819E−11
1.0±1.E−7	0.3469E−7	0.1819E−15
1.0±1.E−9	0.3469E−9	0.1819E−19
1.0±1.E−11	0.3469E−11	0.1818E−23

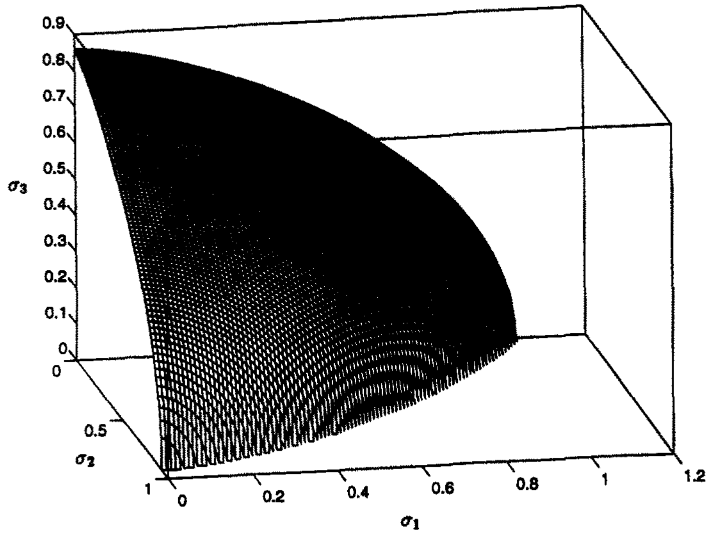

Fig. 1. Shanley–Ryder failure condition surface of eqn (15) for $\sigma_4 = 0.4$.

Figures 1, 2 and 3 show the related deterministic four-dimensional and invariant three-dimensional failure surfaces for this example. The Shanley–Ryder eqn (14) is depicted in Figs 1 and 2 for σ_4 values of 0.4 and 0.6, while Fig. 3 represents the corresponding invariant condition (8) and is the single invariant failure surface for 440,000 combinations of admissible values of the four σ_n satisfying eqn (14). Other plots similar to Figs 1 and 2 can, of course, also be generated each for a distinct single σ_4 value.

In order to remove the possible undesirable effects of odd numbered invariants J_1 and J_3 when the normal stresses vanish, the expressions in eqn (8) were modified to contain the first three even numbered invariants J_k and $\mathscr{J}_k$. This means that in eqns (8), (15), (16) and (17) the k subscripts 1, 2 and 3 are changed to 2, 4 and 6, respectively. The added invariants are

$$J_4 = \tau^i_j\tau^j_k\tau^k_l\tau^l_i \quad J_6 = \tau^i_j\tau^j_k\tau^k_l\tau^l_m\tau^m_n\tau^n_i, \tag{18}$$

with similar expressions for $\mathscr{J}_4$ and $\mathscr{J}_6$ in terms of F^i_j, etc. The invariants J_2 and $\mathscr{J}_2$ were previously defined in eqns (2) and (3). The evaluation of fitting a modified eqn (8), containing these even numbered invariants, to the Shanley–Ryder expression (14) are given in Table

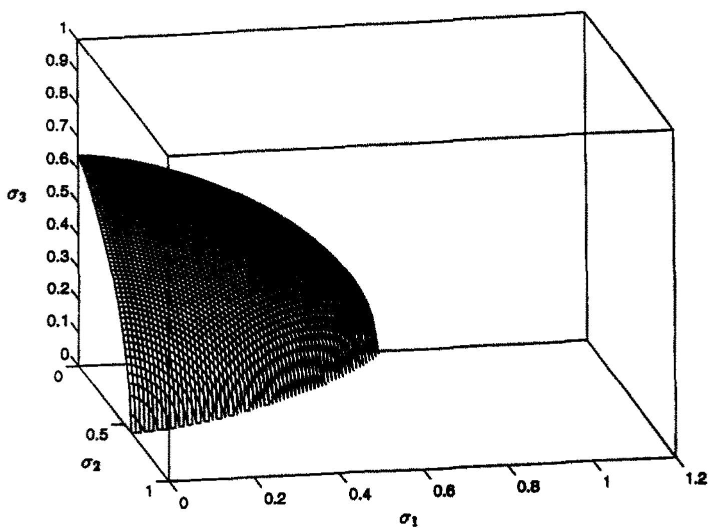

Fig. 2. Shanley–Ryder failure condition surface of eqn (15) for $\sigma_4 = 0.6$.

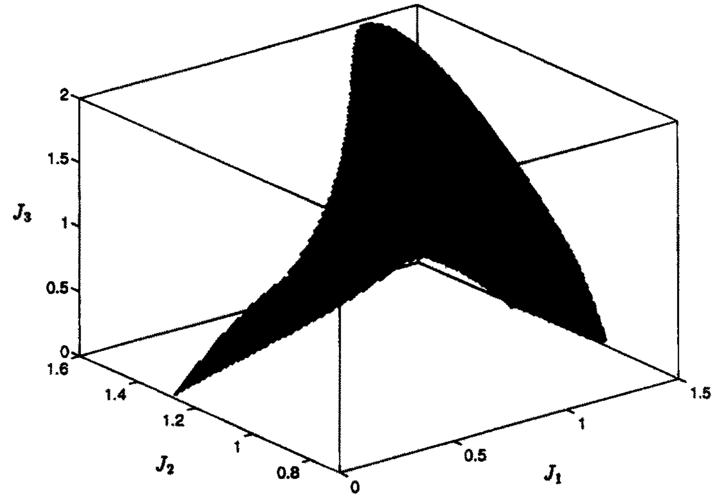

Fig. 3. Invariant failure surface of eqn (8) based on first three invariants.

Table 2. Relation of C in eqn (8) to accuracy of eqn (15) fit for even numbered invariants

C	Maximum fit error	σ^2
1.0±1.E−1	0.2780E−1	0.8999E−4
1.0±1.E−3	0.2605E−3	0.8083E−8
1.0±1.E−5	0.2603E−5	0.8075E−12
1.0±1.E−7	0.2603E−7	0.8075E−16
1.0±1.E−9	0.2603E−9	0.8075E−20
1.0±1.E−11	0.2603−11	0.8073E−24

2. It should be noted that this considerably more nonlinear invariant failure condition, $O((\tau_j^i)^6)$ approximates eqn (14) with the same degree of accuracy as does the original eqn (8) which is based only on the first three invariants, $O((\tau_j^i)^3)$. A plot of the failure condition with even numbered invariants corresponding to the stresses of eqn (14) is shown in Fig. 4. No discernible pattern emerges between the invariant failure surfaces of Figs 3 and 4.

As noted before, the Shanley–Ryder condition cannot be reduced to a single plot for a given set of loading conditions, material properties, etc. For instance, for N combined

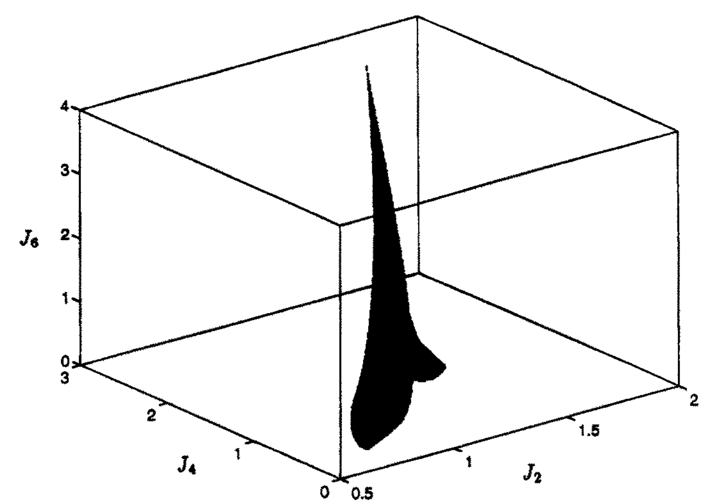

Fig. 4. Invariant failure surface of eqn (8) based on even numbered invariants.

loadings, $N-3$ stresses σ_n need to be parametrically varied to produce multiple three-dimensional plots of the type shown in Figs 1 and 2. On the other hand, a single three-dimensional invariant plot of eqn (8) or its simpler forms depicts all multiple conditions of the above parameters.

The additive invariant failure condition of eqn (4), with either sequentially or odd numbered invariants, could be similarly fitted to eqn (1) using the techniques described by Worthing and Geffner (1943) to solve the nonlinear least squares relations for the exponents b_k in eqn (4). However, the procedure outlined above for the multiplicative failure condition (8) is so much simpler since only linear relations of the type (17) need to be solved for the unknown exponents b_k. Since the accuracy of fit, as shown in Tables 1 and 2, is extremely high, the multiplicative failure surface (8) is extremely satisfactory in characterizing failure under combined deterministic or random loads.

CONCLUSIONS

The two distinct universal invariant failure criteria based on each of three applied and failure stress invariants and three parameters dependent on material properties, geometry and on loading conditions is accurate and representable by a single failure surface regardless of the degree of anisotropy. The three parameters in the multiplicative representation of the failure surface function are relatively easily determined from a set of linear algebraic least square relations. The accuracy of data fit based on expressions with the first three invariants and on those with the first three even numbered invariants is about the same. However, the even invariants, although of considerably higher degree of nonlinearity, do not suffer from possible shortcomings when, under conditions of zero normal stresses, the odd invariants vanish.

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